

# Cardinal Interpolation with Gaussian Kernels <sup>\*†</sup>

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## Abstract

In this paper, interpolation by scaled multi-integer translates of Gaussian kernels is studied. The main result establishes  $L_p$  Sobolev error estimates and shows that the error is controlled by the  $L_p$  multiplier norm of a Fourier multiplier closely related to the cardinal interpolant, and comparable to the Hilbert transform. Consequently, its multiplier norm is bounded independent of the grid spacing when  $1 < p < \infty$ , and involves a logarithmic term when  $p = 1$  or  $\infty$ .

## 1 Introduction

We consider interpolation by means of linear combinations of translates of a fixed Gaussian where the data consists of samples  $f(hk)$ ,  $k$  in  $\mathbb{Z}^n$ , of a continuous functions  $f$  and derive  $L_p$  error estimates in terms of  $h$  and appropriate smoothness properties of  $f$ . Namely, our interpolants are of the form

$$s_h(x) = \sum_{k_0 \in \mathbb{Z}^n} a_k e^{-|x-hk|^2}$$

and, roughly speaking, our estimates are of the form  $\|f - s_h\|_{L_p} \leq Ch^k \|f\|_{W_p^k}$ , valid for sufficiently large  $k$ .

In a series of papers [18, 14, 15, 16], Riemenschneider and Sivakumar have developed a comprehensive theory of cardinal interpolation by Gaussians, treating issues of existence/uniqueness of interpolants, decay of fundamental functions, bounds on Lebesgue constants, and  $L_p$  stability for data in  $\ell_p$ . This is an outgrowth of a general theory of cardinal interpolation that started for univariate splines with Schoenberg [17], was extended to several variables for *box splines* by de

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Boor, Höllig, and Riemenschneider [5], and for radial basis functions by Madych and Nelson [9], and Buhmann[2].

One important topic not addressed by Riemenschneider and Sivakumar is the rate of convergence of the interpolant to a given smooth target function. In general, global approximation of smooth, non-analytic functions with Gaussians and other  $C^\infty$  positive definite functions poses a considerable challenge. In these cases, rates of decay are often known to be *spectral*<sup>1</sup> [10], but typically hold only for target functions that are infinitely smooth. More mainstream, linear error estimates for functions of finite smoothness (from Sobolev spaces, for example), where the  $L_p$  norm of the error decays at a rate governed by the  $L_p$  smoothness of the target function<sup>2</sup> have been more elusive. In this regard, the approximation power of the underlying spaces has only been thoroughly understood in the shift invariant setting, but that of the interpolants has not as yet been studied. In the case of Gaussians, the  $L_2$  error estimates fall under the shift invariant theory developed in [3, 4] and generalized to  $L_p$  by Johnson [7, Section 4].

## 1.1 Overview

In this article, we demonstrate convergence rates for cardinal Gaussian interpolation for target functions having finite smoothness. The basic strategy we adopt has been developed in [13] and is based around a  $K$ -functional argument, usually brought about by band limiting the target function in a precise way. The techniques we use for estimating the error involve showing that interpolating by band-limited functions delivers precise approximation rates, and that such band-limited interpolants form a very useful class of target functions on which cardinal Gaussian interpolation is very stable. In this case, the main challenge is to demonstrate this extra stability; this is accomplished by carefully controlling the multiplier norm of the Lagrange function.

In Section 2, we describe in detail our main result and the strategy that we will use to obtain it. At the end of that section, we discuss a way to generalize the results to other, analogous situations. The approximation results for interpolation by band-limited functions is the focus of Section 3. The main tools for cardinal Gaussian interpolation and the key multiplier estimate are given in Section 4. The extra stability results are demonstrated in Section 5.

## 1.2 Notation and Background

The symbol  $C$ , often with a subscript, will always represent a constant. The subscript is used to indicate dependence on various parameters. The value of  $C$  may change, sometimes within the same line.

Let  $\mathcal{S}$  denote the space of Schwartz functions on  $\mathbb{R}^n$ . The  $n$ -dimensional Fourier transform is given by  $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\langle \xi, x \rangle} dx$ , and its inverse is  $f^\vee(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} f(\xi) e^{i\langle x, \xi \rangle} d\xi$ . An important property of the Gaussian functions

$$g : x \mapsto \exp[-|x|^2], \tag{1.1}$$

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<sup>1</sup>decaying exponentially fast

<sup>2</sup>This stands in contrast to *nonlinear* approximation, where the  $L_p$  norm of the error decays at a rate dependent on a different smoothness norm, but generally Gaussians at different scales must be employed, cf. [6, 8]

is that they satisfy  $\widehat{\mathbf{g}} = \pi^{n/2} \mathbf{g}(\cdot/2)$ .

Given a *multiplier*  $m : \mathbb{R}^n \rightarrow \mathbb{C}$ , which maps  $f$  to  $(\widehat{f} \cdot m)^\vee$  the  $L_p \rightarrow L_p$  operator norm is denoted,

$$\|m\|_{\mathcal{M}_p} := \sup_{\|f\|_p=1} \left\| (\widehat{f} \cdot m)^\vee \right\|_p$$

Let  $\Omega \subseteq \mathbb{R}^n$  be a domain that satisfies a uniform cone condition. The Sobolev space  $W_p^k(\Omega)$  is endowed its usual seminorm  $|\cdot|_{W_p^k(\Omega)}$  and norm  $\|\cdot\|_{W_p^k(\Omega)}$ , defined by

$$|f|_{W_p^k(\Omega)} := \sup_{|\alpha|=k} \|D^\alpha f\|_{L_p(\Omega)}, \quad \text{and} \quad \|f\|_{W_p^k(\Omega)} := \|f\|_{L_p(\Omega)} + |f|_{W_p^k(\Omega)}.$$

When just the seminorm is used, the resulting space is a *Beppo-Levi* space. We will denote it by  $\dot{W}_p^k(\Omega)$ . Finally, most of the time we will be dealing with  $\Omega = \mathbb{R}^n$ . When that is the case, we will just use  $L_p$ ,  $W_p^k$ , or  $\dot{W}_p^k$ .

We denote by  $B(x, r)$  the ball in  $\mathbb{R}^n$  having center  $x$  and radius  $r$ . The space of entire functions of exponential type, viewed as tempered distributions whose Fourier transform is supported in  $B(0, R)$ , is given by

$$\text{PW}(R) := \{f \in \mathcal{S}' \mid \text{supp}(\widehat{f}) \subset B(0, R)\}.$$

This is the *Paley-Wiener space* of band-limited entire functions. In addition, we let  $\text{PW}_p^k(R) := W_p^k \cap \text{PW}(R)$ , the band-limited entire functions in  $W_p^k$ .

## 2 The Main Result

We consider interpolation of a continuous function  $f : C(\mathbb{R}^n) \rightarrow \mathbb{C}$  at gridded centers  $h\mathbb{Z}^n$  using elements of the linear span of shifts of a fixed Gaussian kernel  $g(x) = \exp(-|x|^2)$ , the span being closed in the topology of uniform convergence on compact sets. In other words, we consider interpolation by functions of the form  $s_{f,h}(x) = \sum_{\zeta \in h\mathbb{Z}^n} a_\zeta g(x - \zeta)$ .

The existence and uniqueness of the interpolant is a consequence of the existence/uniqueness of the *Lagrange* function  $\chi_h$ . Indeed,

$$I_h f(x) := \sum_{\xi \in h\mathbb{Z}^n} f(\xi) \chi_h(x - \xi).$$

The Lagrange function is the function in the (extended) span of shifts of  $g$  equaling 1 at the origin and 0 at all other dilated multi-integers. These have been studied in [14, 15], as the *cardinal* interpolant:  $L_\lambda^{[n]}(y) = \sum_{j \in \mathbb{Z}^n} c_j \exp(-\lambda|y - j|^2)$ . The relation between the two is

$$\chi_h(x) = \sum_{\xi \in h\mathbb{Z}^n} b_\xi g(x - \xi) = L_{h^2}^{[n]} \left( \frac{x}{h} \right). \quad (2.1)$$

The problem that we have set forth for ourselves is to obtain a good understanding of how well Gaussian interpolants approximate functions in various Sobolev spaces. Our main result is the following:

**Theorem 2.1 (Main Result).** *Let  $1 < p < \infty$  and  $k > n/p$ . There exists a constant  $C_p$  so that for  $f \in W_p^k(\mathbb{R}^n)$ , the Gaussian interpolant  $I_h f = \sum_{\xi \in h\mathbb{Z}^n} f(\xi) \chi_h(\cdot - \xi)$  satisfies*

$$\|f - I_h f\|_p \leq C_p h^k \|f\|_{W_p^k}.$$

*For  $p = 1$  and  $k \geq n$  or  $p = \infty$  and  $k > 0$ , there is a constant  $C$  so that for  $f \in W_p^k(\mathbb{R}^n)$*

$$\|f - I_h f\|_p \leq C(1 + |\log h|)^n h^k \|f\|_{W_p^k}.$$

The strategy for proving the main theorem involves two steps, which we will discuss before we give the proof. The first is showing that interpolation of functions in  $\dot{W}_p^k$  by band-limited functions on  $h\mathbb{Z}^n$  is possible, and that the band-limited interpolants approximate functions in  $\dot{W}_p^k$  very well. This is the content of the Approximation Property described in Lemma 2.2 below. We will prove this in Section 3.

**Lemma 2.2 (Approximation Property).** *Let  $b = \pi + \varepsilon$  with  $0 < \varepsilon < \pi$ . If  $f \in \dot{W}_p^k$  for  $k > n/p$ ,  $1 \leq p \leq \infty$  then given  $h > 0$  there is a function  $\tilde{f}$  satisfying*

$$\tilde{f} \in \text{PW}(b/h) \tag{2.2}$$

*such that*

$$\tilde{f}|_{h\mathbb{Z}^n} = f|_{h\mathbb{Z}^n} \tag{2.3}$$

$$\|f - \tilde{f}\|_{L_p} \leq C h^k \|f\|_{W_p^k} \tag{2.4}$$

$$|\tilde{f}|_{W_p^k} \leq C \|f\|_{W_p^k} \tag{2.5}$$

The second step is to show that Gaussian interpolation is stable on the space  $\text{PW}_p^k(b/h)$ , endowed with the Sobolev norm. We will prove this in Section 5.

**Stable Interpolation Property.** *Let  $k \in \mathbb{N}$ ,  $k > n/p$ . We say that the interpolation operators  $I_h$  satisfy the stable interpolation property on the family  $\text{PW}_p^k(b/h)$  if there is  $Q_p : h \rightarrow (0, \infty)$  so that one has*

$$\|I_h f\|_{W_p^k} \leq Q_p(h) \|f\|_{W_p^k} \quad \text{for all } f \in \text{PW}_p^k(b/h).$$

**Proposition 2.3.** *Let  $1 \leq p \leq \infty$ . If the interpolation operators  $I_h$  satisfy the Stable Interpolation Property on  $\text{PW}_p^k(b/h)$ , then for  $k > n/p$ ,*

$$\|I_h f - f\|_p \leq C(1 + Q_p(h)) h^k \|f\|_{W_p^k}, \quad f \in W_p^k.$$

*Proof.* We note that, by the Stable Interpolation Property,  $\|f - I_h f\|_p \leq \|f - \tilde{f}\|_p + \|\tilde{f} - I_h \tilde{f}\|_p$ , since  $I_h \tilde{f} = I_h f$ . By the Approximation Property, the error  $\|\tilde{f} - f\|_p$  is controlled by  $h^k$ . Thus, the analysis of interpolation error  $\tilde{f} - I_h \tilde{f}$  reduces to investigating its behavior on  $h\mathbb{Z}^n$ , where the error vanishes. An estimate on the size of a smooth function having many zeroes was proved by Madych & Potter [11, Corollary 1]. Employing it, we obtain

$$\|\tilde{f} - I_h \tilde{f}\|_p \leq C h^k |\tilde{f} - I_h \tilde{f}|_{W_p^k} \leq C h^k \left( |\tilde{f}|_{W_p^k} + |I_h \tilde{f}|_{W_p^k} \right).$$

Thus, the interpolation error is controlled entirely by the norm of  $I_h$  as an operator from  $W_p^k$  to  $W_p^k$  for  $k > n/p$ . Invoking the stable interpolation property, we obtain

$$\begin{aligned}\|f - I_h f\|_p &\lesssim \|f - \tilde{f}\|_p + \|\tilde{f} - I_h \tilde{f}\|_p \\ &\lesssim h^k \|f\|_{W_p^k} + h^k \left( \|\tilde{f}\|_{W_p^k} + Q_p(h) \|\tilde{f}\|_{W_p^k} \right) \\ &\lesssim h^k (1 + Q_p(h)) \|f\|_{W_p^k}.\end{aligned}$$

□

*Proof of Theorem 2.1.* Finishing the proof only requires only showing that the Stable Interpolation Property holds, with the appropriate  $Q_p(h)$ . We do this in Lemma 5.4. There we also show that

$$Q_p(h) \leq \begin{cases} C_p & 1 < p < \infty, \\ C(1 + |\log h|)^n & p = 1, \infty. \end{cases}$$

Apart from  $C_p$  being explicitly dependent on  $p$ , the two constants depend on  $n$ ,  $k$ , and the choice of the parameter  $b$ . Using this estimate on  $Q_p$  in Proposition 2.3 then yields the result.

□

**Generalizations** The interpolation problems considered here specifically involve only spaces of band-limited functions and spaces of Gaussians. However, it is worthwhile to broaden the context and describe these problems in a more general way.

Let  $\Xi \subset \Omega$  be a discrete set of points, which we will call *nodes* or *centers*, which will play the role of  $h\mathbb{Z}^n$  above as sites for interpolation. Since  $\Xi$  doesn't have to lie on a grid, we will describe how dense  $\Xi$  is in  $\Omega$  using the *fill distance* or *mesh norm*, which is defined by  $h(\Xi) := \sup_{x \in \mathbb{R}^n} \text{dist}(x, \Xi)$ . Normally, one would not be dealing with all possible  $\Xi$ , but rather with a specific class of sets,  $\mathbf{X}$ .

Suppose that for each  $\Xi$  in  $\mathbf{X}$  there are two spaces of functions,  $\mathcal{F}_\Xi$  and  $\mathcal{G}_\Xi$ . These form families  $\mathcal{F}_\mathbf{X} = \{\mathcal{F}_\Xi \mid \Xi \in \mathbf{X}\}$  and  $\mathcal{G}_\mathbf{X} = \{\mathcal{G}_\Xi \mid \Xi \in \mathbf{X}\}$ . Here,  $\mathcal{F}_\xi$  and  $\mathcal{G}_\Xi$  are analogous to the band-limited functions  $\text{PW}(b/h)$  and the Gaussians, respectively. Both are contained in  $C(\Omega) \cap \dot{W}_p^k(\Omega)$ . In addition, we will assume that for each  $\mathcal{G}_\Xi$  there is an interpolation operator  $I_{\mathcal{G}_\Xi} : C(\Omega) \rightarrow \mathcal{G}_\Xi$ . Finally, we suppose that  $\mathcal{F}_\mathbf{X}$  obeys an Approximation Property and  $\mathcal{G}_\mathbf{X}$  obeys a Stable Interpolation Property with respect to the family  $\mathcal{F}_\mathbf{X}$ . Then a nearly identical proof to the one for Proposition 2.3 will establish this generalized version of that proposition.

**Proposition 2.4.** *Let  $\Omega \subset \mathbb{R}^n$  be a region satisfying a uniform cone condition. For  $1 \leq p \leq \infty$ , let  $\mathbf{X}$  be a collection of discrete subsets  $\Xi \subset \Omega$  and let  $\mathcal{F}_\mathbf{X}$  be a family satisfying the Approximation Property. If the interpolation operators  $I_{\mathcal{G}_\Xi} : C(\Omega) \rightarrow \mathcal{F}_\Xi$  satisfy the Stable Interpolation Property on  $\mathcal{F}_\mathbf{X}$ , then for  $k > n/p$ ,*

$$\|I_{\mathcal{G}_\Xi} f - f\|_p \leq C(1 + Q_p(\Xi)) h^k \|f\|_{W_p^k}, \quad f \in W_p^k(\Omega).$$

### 3 Band-limited Interpolation and the Approximation Property

In this section, we will prove Lemma 2.2, which asserts that interpolation by band-limited functions satisfies the Approximation Property.

*Proof.* Our aim is prove the existence of the interpolant  $\tilde{f}$ , along with the other properties. Let  $\hat{\varphi}(\xi) \in C^\infty$  be such that

- $\text{supp } \hat{\varphi}(\xi) \subset \{\xi: |\xi|_\infty \leq \pi + \varepsilon\}$
- $\hat{\varphi}(\xi) = 1$  if  $|\xi|_\infty \leq \pi - \varepsilon$
- $\sum_{j \in \mathbb{Z}^n} \hat{\varphi}(\xi - 2\pi j) = 1.$

Note that if  $\hat{\varphi}(\xi)$  satisfies the first two conditions, then setting  $\hat{\rho}(\xi) = \frac{\hat{\varphi}(\xi)}{\sum_{j \in \mathbb{Z}^n} \hat{\varphi}(\xi - 2\pi j)}$  defines a function which satisfies all three. Our candidate for  $\tilde{f}$  is the function

$$g(x) := \sum_{j \in \mathbb{Z}^n} f(hj) \varphi\left(\frac{x}{h} - j\right).$$

Clearly  $g$  is in  $\text{PW}(b/h)$  and also  $g|_{h\mathbb{Z}^n} = f|_{h\mathbb{Z}^n}$ , thus (2.2) and (2.3) are valid for  $\tilde{f} = g$ . To see (2.5) and (2.4) also hold, write

$$f = f_0 + f_1 \quad \text{where} \quad \hat{f}_0(\xi) := \hat{f}(\xi) \hat{\varphi}(2h\xi).$$

Define  $g = g_0 + g_1$  accordingly, i.e.,

$$g_0(x) := \sum_{j \in \mathbb{Z}^n} f_0(hj) \varphi\left(\frac{x}{h} - j\right),$$

and  $g_1 := g - g_0$ . Note that (for sufficiently small  $\varepsilon$ )

$$\hat{g}_0(\xi) = \left\{ \sum_{j \in \mathbb{Z}^n} \hat{f}_0\left(\xi - \frac{2\pi j}{h}\right) \right\} \hat{\varphi}(h\xi) = \hat{f}_0(\xi) \hat{\varphi}(h\xi) = \hat{f}_0(\xi)$$

so

$$g_0(x) = f_0(x). \tag{3.1}$$

The Sobolev seminorms of  $g_0$  and  $f_0$  are controlled by the seminorm of  $f$ :

$$|f_0|_{W_p^k} \leq \sup_{|\alpha|=k} \|D^\alpha(f * \rho)\|_{L_p} \leq \|\rho\|_{L^1} |f|_{W_p^k}, \quad 1 \leq p \leq \infty \tag{3.2}$$

where  $\rho(x) = \frac{1}{(2h)^n} \phi(x/2h)$  and  $\|\rho\|_{L^1} = \|\phi\|_{L^1}$  is independent of  $h$ . On the other hand, since  $f_1 = f - f * \rho$ , we have,

$$\|f_1\|_{L^p} = \|f - (f * \rho)\|_p \leq Ch^k |f|_{W_p^k}, \tag{3.3}$$

the usual error from mollification by a band-limited mollifier. What remains is to control the seminorm of  $g_1$  and to bound the error  $\|g_1 - f_1\|_p$ .

For  $|\alpha| = r$  with  $0 \leq r \leq k$ , we have

$$\begin{aligned} D^\alpha g_1(x) &= \sum_{j \in \mathbb{Z}^n} f_1(hj) D^\alpha \left[ \phi \left( \frac{x}{h} - j \right) \right] \\ &= h^{-r} \sum_{j \in \mathbb{Z}^n} f_1(hj) [D^\alpha \phi] \left( \frac{x}{h} - j \right). \end{aligned}$$

When  $p = \infty$ , it follows that

$$\|D^\alpha g_1\|_\infty \leq h^{-r} \sup_{j \in \mathbb{Z}^n} |f_1(hj)| \times \sup_{x \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}^n} \left| [D^\alpha \phi] \left( \frac{x}{h} - j \right) \right| \leq Ch^{-r} \sup_{j \in \mathbb{Z}^n} |f_1(hj)|,$$

since the fact that  $\phi$  is a Schwartz function implies that  $\sup_{y \in \mathbb{R}^n} \sum_{j \in \mathbb{Z}^n} |[D^\alpha \phi](y - j)| \leq C$ .

For  $p < \infty$ , we have

$$\begin{aligned} |D^\alpha g_1(x)|^p &= (h^{-r})^p \left| \sum_{j \in \mathbb{Z}^n} f_1(hj) [D^\alpha \phi] \left( \frac{x}{h} - j \right) \right|^p \\ &\leq h^{-rp} \left\{ \sum_j |f_1(hj)|^p \left| \left( 1 + \left| \frac{x}{h} - j \right| \right)^{2n} [D^\alpha \phi] \left( \frac{x}{h} - j \right) \right|^p \right\} \\ &\quad \times \left\{ \sum_j \left( 1 + \left| \frac{x}{h} - j \right| \right)^{-2np'} \right\}^{p/p'}. \end{aligned}$$

Since  $\phi$  is a Schwartz function,  $\left\| \left( 1 + \left| \frac{x}{h} - j \right| \right)^{2n} [D^\alpha \phi] \left( \frac{x}{h} - j \right) \right\|_{L^p(\mathbb{R}^n)}^p$  is bounded by  $\tilde{C}_r h^n$  where  $h^n$  comes about from the substitution  $y = \frac{x}{h} - j$  in the integration over  $\mathbb{R}^n$ . Consequently,

$$\|D^\alpha g_1\|_{L^p} \leq h^{-r} \tilde{C}_r \left\{ \sum |f_1(hj)|^p h^n \right\}^{1/p} \times \max_{x \in \mathbb{R}^n} \left\{ \sum_j \left( 1 + \left| \frac{x}{h} - j \right| \right)^{-2np'} \right\}^{p/p'}$$

The sequence  $j \mapsto \left( 1 + \left| y - j \right| \right)^{-2n}$  is bounded in  $\ell_p$ , uniformly for all  $y \in \mathbb{R}^n$ , by a constant  $C$ . Thus, for  $1 \leq p \leq \infty$ ,

$$\begin{aligned} |g_1|_{W_p^r} &\leq C_r h^{-r} h^{n/p} \|f_1\|_{h\mathbb{Z}^n} \|f_1\|_{\ell_p(\mathbb{Z}^n)} \\ &\leq C_r h^{-r} \left[ \|f_1\|_{L_p} + h^k |f_1|_{W_p^k} \right]. \end{aligned} \tag{3.4}$$

This follows by scaling the estimate  $\|F\|_{\ell_p(\mathbb{Z}^n)} \leq C(\|F\|_{L_p(\mathbb{R}^n)} + |F|_{W_p^k(\mathbb{R}^n)})$ , which holds for  $k > n/p$  (this is a simple consequence of the Sobolev embedding theorem). By applying (3.2) and (3.3), we obtain

$$|g_1|_{W_p^r} \leq Ch^{k-r} |f|_{W_p^k}.$$

Finally, we have

$$|g|_{W_p^k} = |g_0 + g_1|_{W_p^k} = |f_0 + g_1|_{W_p^k} \leq |f_0|_{W_p^k} + |g_1|_{W_p^k} \leq C|f|_{W_p^k}$$

which follows from (3.2) and above. So (2.5) is established for  $1 \leq p \leq \infty$ . Also from (3.1), (3.2) and (3.4) we get

$$\begin{aligned}\|f - g\|_{L^p} &= \|f_1 - g_1\|_{L^p} \leq \|f_1\|_{L^p} + \|g_1\|_{L^p} \\ &\leq Ch^k |D^k f|_{L^p}.\end{aligned}$$

and (2.4) holds for  $1 \leq p \leq \infty$ . Hence, it follows that  $g$  is indeed  $\tilde{f}$ .  $\square$

## 4 Cardinal Interpolation with Gaussians

We now investigate the Fourier transform of the Lagrange function, and, through this, the Fourier transform of the gridded Gaussian interpolant. Since  $\widehat{\chi_h}(\xi) = h^n \widehat{L_\lambda^{[n]}}(h \cdot \xi)$  with  $\lambda = h^2$ , and because

$$\widehat{L_\lambda^{[n]}}(\omega) = \frac{\exp(-|\xi|^2/(4\lambda))}{\sum_{k \in \mathbb{Z}^n} \exp(-(|\xi - 2\pi k|^2)/(4\lambda))}$$

we have

$$\widehat{\chi_h}(\xi) = h^n \frac{\exp(-\frac{1}{4}|\xi|^2)}{\sum_{k \in \mathbb{Z}^n} \exp(-\frac{1}{4}|\xi - \frac{2\pi k}{h}|^2)} =: h^n m_h^{[n]}(\xi). \quad (4.1)$$

Throughout the rest of this article, the multiplier  $m_h^{[n]}$  is the subject of much of our investigation. Its multiplier norm controls the the stability of the interpolation process on Sobolev spaces, and ultimately, it determines the rate of decay of interpolation error. Because it is, evidently, a  $n$ -fold tensor product of univariate multipliers:  $m_h^{[n]}(\xi) = m_h^{[1]}(\xi_1) \cdots m_h^{[1]}(\xi_n)$ , many of our results can be obtained by considering the 1 dimensional case, where we suppress the dimension to write  $m_h := m_h^{[1]}$ .

Finally, formula (4.1) allows us to easily express the Fourier transform of the interpolant:

$$\begin{aligned}\widehat{I_h f}(\xi) &= \left[ \sum f(hj) \chi_h(\cdot - hj) \right]^\wedge(\xi) = h^n \left[ \sum_{j \in \mathbb{Z}^n} f(hj) e^{-i\langle hj, \xi \rangle} \right] m_h^{[n]}(\xi) \\ &= \left[ \sum_{\beta \in 2\pi\mathbb{Z}^n} \widehat{f}\left(\xi - \frac{\beta}{h}\right) \right] m_h^{[n]}(\xi).\end{aligned} \quad (4.2)$$

(The final equality follows from Poisson's summation formula.)

### 4.1 The univariate multiplier $m_h$

In the interest of keeping results self contained, we now provide some simple estimates on the size of the multiplier. Such estimates (and stronger ones) could, with some effort, be gleaned from the work of Riemenschneider and Sivakumar. However, the ones we provide here are easier to obtain than those presented in [15, Theorem 3.3 and Corollary 3.4], yet totally sufficient for our purposes.



We proceed in two stages. In the first stage we obtain estimates on  $m_h^\vee$  that hold independently of  $h$ , but are rather slow (in particular, they do not show that  $m_h^\vee$  is integrable). In the second stage, we demonstrate that  $m_h^\vee(\xi)$  decays like  $\mathcal{O}(|x|^{-2})$ , and is, hence, integrable, but these estimates depend strongly on  $h$ .

**First estimate of  $m_h^\vee$ :** Since  $m_h(\xi) > 0$ , we have  $|m_h^\vee(x)| \leq m_h^\vee(0) = 1$ . On the other hand, from (4.1),

$$m_h(\xi) = \left( \sum_{k \in \mathbb{Z}} e^{-\frac{\pi^2 k^2}{h^2}} e^{-\frac{\pi k \xi}{h}} \right)^{-1} = \left( 1 + 2 \sum_{k=0}^{\infty} e^{-\frac{\pi^2 k^2}{h^2}} \cosh(\pi k \xi / h) \right)^{-1} =: (d_0(\xi))^{-1}.$$

Hence  $m'_h(\xi) = -(m_h(\xi))^2 d_1(\xi)$  and  $m''_h(\xi) = 2(m_h(\xi))^3 (d_1(\xi))^2 - (m_h(\xi))^2 n_2(\xi)$ , where we have defined  $d_1(\xi) := d'_0(\xi)$  and  $d_2 := n'_1(\xi)$ . Hence,

$$d_1(\xi) = 2 \sum_{k=1}^{\infty} \left( \frac{\pi k}{h} \right) e^{-\frac{\pi^2 k^2}{h^2}} \sinh \left( \frac{\pi k \xi}{h} \right) \quad \text{and} \quad d_2(\xi) = 2 \sum_{k=1}^{\infty} \left( \frac{\pi k}{h} \right)^2 e^{-\frac{\pi^2 k^2}{h^2}} \cosh \left( \frac{\pi k \xi}{h} \right).$$

It follows that  $m'_h(\xi) < 0$  for positive  $\xi$ , and, by symmetry,  $m'_h(\xi) > 0$  for negative  $\xi$ . Therefore,  $\int_0^\infty |m'_h(\xi)| d\xi = -\int_0^\infty m'_h(\xi) d\xi = m_h(0) < 1$ . It follows that

$$\|m'_h\|_1 < 2, \quad (4.3)$$

which implies that  $|m_h^\vee(x)| < 2/|x|$ . Thus,

$$|m_h^\vee(x)| \leq \min(1, 2/|x|). \quad (4.4)$$

**Second estimate of  $m_h^\vee$ :** Using some simple algebraic manipulations, we may rewrite the second derivative as

$$m''_h(\xi) = m_h(\xi) \left[ 2 \left( \frac{\sum_{k \in \mathbb{Z}} \left( \frac{k\pi}{h} \right) e^{-\frac{1}{4}|\xi - \frac{2\pi k}{h}|^2}}{\sum_{k \in \mathbb{Z}} e^{-\frac{1}{4}|\xi - \frac{2\pi k}{h}|^2}} \right)^2 - \left( \frac{\sum_{k \in \mathbb{Z}} \left( \frac{k\pi}{h} \right)^2 e^{-\frac{1}{4}|\xi - \frac{2\pi k}{h}|^2}}{\sum_{k \in \mathbb{Z}} e^{-\frac{1}{4}|\xi - \frac{2\pi k}{h}|^2}} \right) \right] \quad (4.5)$$

This leads us to the the following estimate:

**Lemma 4.1.** *Let  $h \leq 1$ . There exists a constant  $C$  so that for  $\tilde{k} = 1, 2, \dots$ , and  $\xi \in [\frac{2\pi(\tilde{k}-1)}{h}, \frac{2\pi\tilde{k}}{h}]$ ,*

$$|m''_h(\xi)| \leq C \left( \frac{\tilde{k}}{h} \right)^2 m_h(\xi).$$

*Proof.* We write

$$I := \left( \frac{\sum_{k \in \mathbb{Z}} \left( \frac{k\pi}{h} \right) e^{-\frac{1}{4}|\xi - \frac{2\pi k}{h}|^2}}{\sum_{k \in \mathbb{Z}} e^{-\frac{1}{4}|\xi - \frac{2\pi k}{h}|^2}} \right)^2 \quad \text{and} \quad II := \left( \frac{\sum_{k \in \mathbb{Z}} \left( \frac{k\pi}{h} \right)^2 e^{-\frac{1}{4}|\xi - \frac{2\pi k}{h}|^2}}{\sum_{k \in \mathbb{Z}} e^{-\frac{1}{4}|\xi - \frac{2\pi k}{h}|^2}} \right)$$

We split the numerator of  $I$  to isolate its two principal terms

$$\left| \sum_{k \in \mathbb{Z}} \left( \frac{k\pi}{h} \right) e^{-\frac{1}{4}|\xi - \frac{2\pi k}{h}|^2} \right| \leq \left( \frac{\tilde{k}\pi}{h} \right) \left( e^{-\frac{1}{4}|\xi - \frac{2\pi(\tilde{k}-1)}{h}|^2} + e^{-\frac{1}{4}|\xi - \frac{2\pi\tilde{k}}{h}|^2} \right) + 2 \sum_{k=1}^{\infty} \left( \frac{k\pi}{h} + \frac{\tilde{k}\pi}{h} \right) e^{-|\frac{\pi k}{h}|^2}.$$

Since it is a series of nonnegative terms, the denominator can be bounded below by its two largest summands:  $\sum_{k \in \mathbb{Z}} e^{-\frac{1}{4}|\xi - \frac{2\pi k}{h}|^2} \geq e^{-\frac{1}{4}|\xi - \frac{2\pi(\tilde{k}-1)}{h}|^2} + e^{-\frac{1}{4}|\xi - \frac{2\pi\tilde{k}}{h}|^2} \geq 2e^{-\frac{\pi^2}{4h^2}}$ . Therefore,

$$I \leq \left[ \left( \frac{\tilde{k}\pi}{h} \right) + \sum_{k=1}^{\infty} \left( \frac{k\pi}{h} + \frac{\tilde{k}\pi}{h} \right) e^{-\left( \left| \frac{\pi k}{h} \right|^2 - \frac{\pi^2}{4h^2} \right)} \right]^2.$$

A similar argument shows that

$$II \leq \left( \frac{\tilde{k}\pi}{h} \right)^2 + \sum_{k=1}^{\infty} \left( \frac{k\pi}{h} + \frac{\tilde{k}\pi}{h} \right)^2 e^{-\left( \left| \frac{\pi k}{h} \right|^2 - \frac{\pi^2}{4h^2} \right)}.$$

□

An immediate consequence is that for  $h \leq 1$ ,  $\|m_h''\|_{L_1} \leq Ch^{-3}$ . Indeed, one may estimate the integral on an interval around the origin  $[-2\pi/h, 2\pi/h]$  and along the punctured real line  $\Omega := \mathbb{R} \setminus [-2\pi/h, 2\pi/h]$  to obtain

$$\int_{-2\pi/h}^{2\pi/h} |m''(\xi)| d\xi \leq C/h^3$$

and

$$\int_{\Omega} |m''(\xi)| d\xi \leq 2 \sum_{k=2}^{\infty} \frac{2\pi}{h} \|m''\|_{L_{\infty}([ \frac{2\pi(k-1)}{h}, \frac{2\pi k}{h} ])} \leq 2C \sum_{k=2}^{\infty} \frac{2\pi}{h} \left( \frac{k}{h} \right)^2 e^{-\left( \left| \frac{\pi k}{h} \right|^2 - \frac{\pi^2}{4h^2} \right)} \leq C.$$

It follows that

$$|m_h^{\vee}(x)| \leq \frac{C}{h^3|x|^2}. \quad (4.6)$$

## 4.2 The multiplier norm of $m_h^{[n]}$

In the Section 5 we investigate the Sobolev stability of Gaussian interpolation on spaces of band-limited functions. Of particular importance are the operator norms of the convolution operators with Fourier multiplier  $m_h^{[n]}$ . These can be estimated in the case  $p = 1$  and  $\infty$  by using the bounds on the cardinal interpolant obtained in [15, Section 3] (although the ones developed in the previous subsection, (4.4) and (4.6), suffice). In case  $1 < p < \infty$ , multiplier norms are estimated by appealing directly to the formula (4.1), and making a comparison to the Hardy–Littlewood maximal function and the (maximal) Hilbert transform (this is a continuous version of an idea used in [12, Theorem 3.1]).

**Lemma 4.2.** *For  $1 < p < \infty$ , there is a constant  $C_p$  so that*

$$\|m_h^{[n]}\|_{\mathcal{M}_p} \leq C_p.$$

*For  $p = 1, \infty$  there is a constant  $C$  so that*

$$\|m_h^{[n]}\|_{\mathcal{M}_1} = \|m_h^{[n]}\|_{\mathcal{M}_{\infty}} \leq C(1 + |\log h|)^n.$$

*Proof.* Because  $m_h^{[n]}$  is a tensor product of univariate multipliers, it suffices to consider the case  $n = 1$ .

For  $1/p + 1/p' = 1$ , we have  $\|m_h\|_{\mathcal{M}_p} = \|m_h\|_{\mathcal{M}_{p'}}$ . Thus,  $\|m_h\|_{\mathcal{M}_1} = \|m_h\|_{\mathcal{M}_\infty} \leq \|m_h^\vee\|_1$ , by Hölder's inequality. From (4.4) and (4.6), we have

$$\begin{aligned} \int_{\mathbb{R}} |m_h^\vee(x)| dx &\leq 2 \left[ 1 + 2 \int_1^{h^{-3}} (x)^{-1} dx + Ch^{-1} \int_{h^{-3}}^\infty (hx)^{-2} dx \right] \\ &= [C + 12|\log h|] \leq C(1 + |\log h|) \end{aligned}$$

We now consider the case  $1 < p < \infty$ . Let  $f \in L_p$  and let  $g \in L_{p'}$  with  $\|f\|_p = 1 = \|g\|_{p'}$ . We can estimate  $\|m_h\|_{\mathcal{M}_p}$  by the supremum of the expression  $|\int_{\mathbb{R}} \int_{\mathbb{R}} f(x) m_h^\vee(t-x) g(t) dx dt|$ . To this end, let  $\Omega_h(t) := \mathbb{R} \setminus (t-h, t+h)$ . Then,

$$\begin{aligned} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} f(x) m_h^\vee(t-x) g(t) dx dt \right| &\leq \left| \int_{\mathbb{R}} \int_{t-h}^{t+h} f(x) m_h^\vee(t-x) g(t) dx dt \right| \\ &\quad + \left| \int_{\mathbb{R}} \int_{\Omega_h(t)} f(x) m_h^\vee(t-x) g(t) dx dt \right| \\ &=: I + II \end{aligned}$$

The first integral  $I \leq \int_{\mathbb{R}} h^{-1} \int_{t-h}^{t+h} |f(x)| |g(t)| dx dt$ , can be compared to an integral involving the maximal function of  $f$ ,  $f^\sharp(t) := \sup_{\epsilon > 0} (2\epsilon)^{-1} \int_{t-\epsilon}^{t+\epsilon} |f(x)| dx$ . Thus,

$$I \leq 2 \int_{\mathbb{R}} f^\sharp(t) |g(t)| dt \leq 2 \|f^\sharp\|_p \|g\|_{p'} \leq 2C_p \|f\|_p \|g\|_{p'}.$$

To treat  $II$ , we note that,  $m_h'(\xi)$  is integrable, and  $II$  can be estimated by employing the Fourier transform of  $m_h^\vee$ , (this is a trick similar to the one used in [12]):

$$\begin{aligned} II &= \left| \int_{\mathbb{R}} \int_{\Omega_h(t)} \int_{\mathbb{R}} f(x) m_h(\xi) e^{i(t-x)\xi} g(t) d\xi dx dt \right| \\ &= \left| \int_{\mathbb{R}} \int_{\Omega_h(t)} \int_{\mathbb{R}} \frac{f(x)}{i(x-t)} m_h'(\xi) e^{i(t-x)\xi} g(t) d\xi dx dt \right| \\ &= \left| \int_{\mathbb{R}} m_h'(\xi) \int_{\mathbb{R}} \int_{\Omega_h(t)} \frac{f(x)}{i(x-t)} e^{i(t-x)\xi} g(t) dx dt d\xi \right|. \end{aligned}$$

The second equality follows by integration by parts, while the third follows by Fubini's theorem, since  $m_h'$  is integrable on  $\mathbb{R}$  and  $\frac{f(x)}{(x-t)} g(t) \mathbf{1}_{\Omega_t}(x)$  is integrable on  $\mathbb{R}^2$ . It follows that

$$\begin{aligned} II &\leq \int_{\mathbb{R}} |m_h'(\xi)| \int_{\mathbb{R}} \left| \int_{\Omega_h(t)} \frac{e^{-i\xi x} f(x)}{x-t} dt \right| |g(t)| dt d\xi \\ &\leq \int_{\mathbb{R}} |m_h'(\xi)| \int_{\mathbb{R}} [\mathbf{H}(e^{-i\xi \cdot} f)](t) |g(t)| dt d\xi \\ &\leq C_p \|m_h'\|_1 \|f\|_p \|g\|_{p'} \leq 2C_p \|f\|_p \|g\|_{p'}. \end{aligned}$$

In the second inequality, we use the *maximal Hilbert transform*  $[\mathbf{H}F](x) := \sup_{\epsilon>0} |\int_{\Omega_\epsilon(t)} F(t) \frac{dt}{x-t}|$ . The third inequality follows from the fact that the maximal Hilbert transform is strong type  $(p, p)$  (i.e.,  $\|\mathbf{H}F\|_p \leq C_p \|F\|_p$ ) for  $1 < p < \infty$ , as observed in [1, Theorem 4.9]. The fourth inequality follows from (4.3).  $\square$

## 5 Stable interpolation of band-limited functions by Gaussians

We consider in this section interpolation of functions in  $\text{PW}(b/h)$  and show that Gaussian interpolation restricted to such functions is stable with respect to each Sobolev norm  $W_p^k$ , with  $k > n/p$ .

Before stating and proving the result in its full generality, we indicate how the proof works in the univariate case. We use (4.2) to write, for a nonnegative integer  $k$ ,

$$(\widehat{I_h f})^{(j)}(\xi) = \xi^j \left[ \sum_{\beta \in 2\pi\mathbb{Z}} \widehat{f}(\xi - \frac{\beta}{h}) \right] m_h(\xi) = \sum_{\beta \in 2\pi\mathbb{Z}} \widehat{G_{j,\beta}}(\xi)$$

where  $G_{j,\beta} := D^j [(e^{i(\cdot)\beta/h}) f] * (m_h)^\vee$ . Clearly,  $\|I_h f\|_{W_p^k} \leq \sum_{j=0}^k \sum_{\beta} \|G_{j,\beta}\|_p$ .

The result we are after requires estimating  $\|G_{j,\beta}\|_p$  for various values of  $\beta$ . These estimates fall into three categories:  $\beta = 0$ ,  $|\beta| = 2\pi$ , and  $|\beta| > 2\pi$ .

**Estimating  $\|G_{j,0}\|_p$ :** In this case,  $\|D^j f * (m_h)^\vee\|_p \leq \|D^j f\|_p \|m_h\|_{\mathcal{M}_p}$  from which we obtain  $\sum_{j \leq k} \|G_{j,0}\|_p \leq C \|f\|_{W_p^k} \|m_h\|_{\mathcal{M}_p}$ . Thus, from Lemma 4.2,

$$\sum_{j \leq k} \|G_{j,0}\|_p \leq \begin{cases} C_p \|f\|_{W_p^k} & 1 < p < \infty \\ C(1 + |\log h|) \|f\|_{W_p^k} & p = 1, \infty \end{cases} \quad (5.1)$$

where  $C$  and  $C_p$  depend only on  $k$  and  $p$ .

For some of the other terms, we need to use a special cutoff function. We first consider a smooth bump function,  $v$ . It is a non-negative  $C^\infty$  function satisfying

- $v(\xi) = 0$  if  $|\xi| > 2\epsilon$  and  $v(\xi) = 1$  if  $|\xi| < \epsilon$ ,
- $v(-\xi) = v(\xi)$ .

We use  $v$  to construct univariate cutoff functions  $\varphi$  having support in  $[-\pi - 2\epsilon, \pi + 2\epsilon]$ :

- $\varphi(t) = 1$  for  $-\pi \leq t \leq \pi$ ;
- $\varphi(-t) = \varphi(t) = v(t - \pi)$  for  $t \geq \pi$ .

**Estimating  $\|G_{j,\beta}\|_p$ , for  $|\beta| > 2\pi$ :** In this case, the fact that  $f$  is band limited,  $\text{supp } \hat{f} \subset B(0, (\pi + \epsilon)/h)$ , allows us to write

$$\widehat{G_{j,\beta}}(\xi) = \xi^j \widehat{f}(\xi - \frac{\beta}{h}) m_h(\xi) = \widehat{f}(\xi - \frac{\beta}{h}) \xi^j \varphi(h(\xi - \frac{\beta}{h})) m_h(\xi).$$

The norm of the multiplier  $\tau_1 := \tau_{1,j,\beta}(\xi) := \xi^j \varphi(h(\xi - \frac{\beta}{h})) m_h(\xi)$  can be estimated by  $\|\tau_1\|_1$ , which we can estimate using the fact that

$$\max \left( \int_{\mathbb{R}} \left| \frac{d^2 \widehat{g}}{d\xi^2}(\xi) \right| d\xi, \int_{\mathbb{R}} |\widehat{g}(\xi)| d\xi \right) \leq K \implies |g(x)| \leq \frac{K}{(1+|x|)^2} \implies \|g\|_1 \leq CK.$$

By (5.9) of Lemma 5.5, we have the following.

**Claim 5.1.** *For  $\beta \in 2\pi\mathbb{Z}^n$ ,  $|\beta| > 2\pi$ ,  $\|\tau_{1,j,\beta}\|_1 \leq (C/h)|\beta/h|^{(2+j)} \exp(-c|\beta|^2/h^2)$ .*

Therefore,

$$\|G_{j,\beta}\|_p \leq \|\tau_1\|_1 \left\| \left( \widehat{f}(\xi - \frac{\beta}{h}) \right)^\vee \right\|_p \leq \left( \frac{C}{h} \right) \left| \frac{\beta}{h} \right|^{(2+j)} \exp \left( -c \frac{|\beta|^2}{h^2} \right) \|f\|_p. \quad (5.2)$$

**Estimating  $\|G_{j,\beta}\|_p$ , for  $|\beta| = 2\pi$ :** This remaining case is quite similar to the previous one. We again use the fact that  $f$  is band limited, although we need to exercise extra caution since the cutoff  $\varphi(h(\xi - \beta/h))$  overlaps a narrow region (near to  $\beta/2h$ ) where  $m_h(\xi)$  is close to 1 and  $|\xi^j|$  is very large.

When  $\beta = 2\pi$  we write  $\varphi(t) = \omega(t) + v(t + \pi)$ , with  $\omega$  having support on the (non-symmetric interval)  $[-\pi + \epsilon, \pi + 2\epsilon]$  and equaling 1 on  $[-\pi + 2\epsilon, \pi + \epsilon]$ . (When  $\beta = -2\pi$  an obvious modification  $\varphi(t) = \tilde{\omega}(t) + v(t + \pi)$  is made.) This allows us to write

$$\begin{aligned} \widehat{G_{j,\beta}}(\xi) &= \widehat{f}(\xi - \frac{\beta}{h}) \xi^j v(h(\xi - \frac{\pi}{h})) m_h(\xi) + \widehat{f}(\xi - \frac{\beta}{h}) \xi^j \omega(h(\xi - 2\pi/h)) m_h(\xi) \\ &=: \widehat{f}(\xi - \frac{\beta}{h}) \tau_{2,j,\beta}(\xi) + \widehat{f}(\xi - \frac{\beta}{h}) \tau_{3,j,\beta}(\xi). \end{aligned} \quad (5.3)$$

We first investigate  $\tau_2 = \tau_{2,j,\beta}$  by rewriting the monomial  $\xi^j$  as a Taylor series about  $2\pi/h$ , obtaining

$$\begin{aligned} \tau_2(\xi) &= \xi^j v(h(\xi - \frac{\pi}{h})) m_h(\xi) = \sum_{\ell=0}^j \binom{j}{\ell} \left( \frac{2\pi}{h} \right)^{j-\ell} \left( \xi - \frac{2\pi}{h} \right)^\ell v(h(\xi - \frac{\pi}{h})) m_h(\xi) \\ &= \left[ \left( \xi - \frac{2\pi}{h} \right)^j \right] \times m_h(\xi) \times \left[ \sum_{\ell=0}^j \binom{j}{\ell} \left( \frac{2\pi}{h} \right)^{j-\ell} \frac{v(h(\xi - \frac{\pi}{h}))}{(\xi - \frac{2\pi}{h})^{j-\ell}} \right]. \end{aligned}$$

The multiplier norm of  $\mu(\xi) := \sum_{\ell=0}^j \binom{j}{\ell} \left( \frac{2\pi}{h} \right)^{j-\ell} \frac{v(h(\xi - \frac{\pi}{h}))}{(\xi - \frac{2\pi}{h})^{j-\ell}} = \sum_{\ell=0}^j \binom{j}{\ell} (2\pi)^{j-\ell} \frac{v(h\xi - \pi)}{(h\xi - 2\pi)^{j-\ell}}$  (which is a Schwarz function, since the support of  $v(\cdot - \pi)$  is positive distance from  $2\pi$ ) is uniformly bounded (in  $h$ ) by the constant

$$\mathcal{K}_j := \sum_{\ell=0}^j \binom{j}{\ell} (2\pi)^{j-\ell} \left\| \left( \frac{v(\cdot - \pi)}{(\cdot - 2\pi)^{j-\ell}} \right)^\vee \right\|_1.$$

Thus we have shown the following:

**Claim 5.2.** *The multiplier  $\tau_{2,j,\beta}(\xi)$  can be written  $\tau_{2,j,\beta}(\xi) = (\xi - \beta/h)^j \times m_h(\xi) \times \mu(\xi)$ , where  $\mu$  has multiplier norm  $\|\mu\|_{\mathcal{M}_p} \leq \mathcal{K}_j$ , bounded independent of  $h$ .*

The multiplier  $\tau_{3,j,\beta}$  is controlled in a similar way to  $\tau_1$ . The only modification is that we use estimate (5.11) of Lemma 5.5, utilizing the fact that  $\omega(\cdot - 2\pi)$  has support a positive distance (namely  $\epsilon$ ) from  $\pi$ . In particular, it satisfies condition (5.10). Thus, we obtain

$$\|(\tau_3)^\vee\|_1 \leq \left| \frac{C}{h} \right|^{j+3} e^{-|c|^2/h^2},$$

and we have demonstrated the following claim.

**Claim 5.3.** *Let  $|\beta| = 2\pi$ . There is a constant  $C$  depending only on  $j$  for which the multiplier  $\tau_{3,j,\beta}$  satisfies*

$$\|(\tau_{3,j,\beta})^\vee\|_1 \leq C.$$

For  $|\beta| = 2\pi$ , applying the Claims 5.2 and 5.3 to (5.3), it follows that

$$\begin{aligned} \|G_{j,\beta}\|_p &\leq \|(\tau_2 \widehat{f}(\cdot - \beta/h))^\vee\|_p + \|\tau_3 \widehat{f}(\cdot - \beta/h)^\vee\|_p \\ &\leq \mathcal{K}_j \|m_h\|_{\mathcal{M}_p} \left\| \left( (\cdot - \beta/h)^j \widehat{f}(\cdot - \beta/h) \right)^\vee \right\|_p + C \left\| \left( \widehat{f}(\cdot - \beta/h) \right)^\vee \right\|_p \\ &\leq \mathcal{K}_j \|m_h\|_{\mathcal{M}_p} \|f\|_{W_p^j} + C \|f\|_p \end{aligned} \quad (5.4)$$

Summing  $\|G_{j,\beta}\|_p$  over  $0 \leq j \leq k$  and  $\beta \in 2\pi\mathbb{Z}$ , and employing the estimates (5.1), (5.2) and (5.4), we observe that  $\|I_h f\|_{W_p^k} \leq C(1 + \|m_h\|_{\mathcal{M}_p}) \|f\|_{W_p^k}$  for  $f \in \text{PW}((\pi + \epsilon)/h)$ .

We now give the general, multivariate result.

**Lemma 5.4.** *Let  $0 < \epsilon < \pi/2$ . Cardinal interpolation by Gaussians on  $h\mathbb{Z}^n$  restricted to band-limited functions in  $\text{PW}((\pi + \epsilon)/h)$ , satisfies the Stable Interpolation Property, with*

- $Q_p(h) \leq C_p$ , a constant depending only on  $\epsilon$ ,  $p$ ,  $n$  and  $k$  when  $1 < p < \infty$ ,
- $Q_p(h) \leq C(1 + |\log h|)^n$ , a constant depending only on  $\epsilon$ ,  $n$  and  $k$  when  $p = 1, \infty$ ,

In other words,

$$\|I_h f\|_{W_p^k} \leq Q_p(h) \|f\|_{W_p^k} \quad \text{for } f \in \wp\left(\frac{\pi + \epsilon}{h}\right)$$

*Proof.* We use (4.2) to write, for a multi-integer  $|\alpha| \leq k$ ,

$$\widehat{D^\alpha I_h f}(\xi) = \xi^\alpha \left[ \sum_{\beta \in 2\pi\mathbb{Z}^n} \widehat{f}\left(\xi - \frac{\beta}{h}\right) \right] m_h^{[n]}(\xi) = \sum_{\beta \in 2\pi\mathbb{Z}^n} \widehat{G_{\alpha,\beta}}(\xi)$$

where  $G_{\alpha,\beta} := D^\alpha \left[ (f e^{i(\beta/h, \cdot)}) * (m_h^{[n]})^\vee \right]$ . Clearly,  $\|I_h f\|_{W_p^k} \leq \sum_{|\alpha| \leq k} \sum_{\beta} \|G_{\alpha,\beta}\|_p$ , and the remainder of the section is concerned with estimating  $\|G_{\alpha,\beta}\|_p$  for various  $\beta$ .

We write  $\widehat{G_{\beta,\alpha}}(\xi) = M_\alpha(\xi) \widehat{f}(\xi - \beta/h)$  employing the tensor product multiplier  $M_\alpha(\xi) := \xi^\alpha m_h^{[n]}(\xi)$ . We estimate these terms by observing that the support of  $\widehat{f}(\xi - \beta/h)$ , which is a neighborhood of  $\beta/h$ , and therefore lies in a region where  $\mu_\alpha$  is small (for most values of  $\beta \neq 0$ ).

This heuristic is complicated by the fact that, for certain values of  $\beta$ , the neighborhood of  $\beta/h$  overlaps a region where  $m_h^{[n]}$  is near to 1 and  $|\xi^\alpha|$  may be quite large. Therefore, we must be somewhat careful.

Fix  $\beta = (\beta_1, \dots, \beta_n) \in 2\pi\mathbb{Z}^n$ . Before proceeding, we partition  $(1, \dots, n)$  into three subsequences  $(J_1, \dots, J_{n_1})$ ,  $(K_1, \dots, K_{n_2})$  and  $(L_1, \dots, L_{n_3})$ , with  $n_1 + n_2 + n_3 = n$ , and where

- $(J_1, \dots, J_{n_1})$  are the indices  $J$  where  $|\beta_J| > 2\pi$
- $(K_1, \dots, K_{n_2}) = \text{supp}(\beta)$ ,
- $(L_1, \dots, L_{n_3})$  are the indices where  $|\beta_L| = 2\pi$ .

As an example, in dimension  $n = 5$ , we have for  $\beta = (-4\pi, -2\pi, 0, 0, 6\pi)$  that  $(J_1, J_2) = (1, 5)$ ,  $(K_1, K_2) = (3, 4)$  and  $L_1 = 2$ .

Then, because the multiplier  $m_h^{[n]}$  and the monomial  $\xi^\alpha$  are tensor products and can be factored over the three subsequences we have just constructed, and because  $f$  is band limited, we have

$$\begin{aligned} M_\alpha(\xi) \widehat{f}(\xi - \frac{\beta}{h}) &= \left[ \prod_{j=1}^{n_1} (\xi_{J_j})^{\alpha_{J_j}} \varphi(h(\xi_{J_j} - \frac{\beta_{J_j}}{h})) m_h(\xi_{J_j}) \right] \\ &\quad \times \left[ \prod_{j=1}^{n_2} (\xi_{K_j})^{\alpha_{K_j}} m_h(\xi_{K_j}) \right] \\ &\quad \times \left[ \prod_{j=1}^{n_3} (\xi_{L_j})^{\alpha_{L_j}} \varphi(h(\xi_{L_j} - \frac{\beta_{L_j}}{h})) m_h(\xi_{L_j}) \right] \times \widehat{f}(\xi - \frac{\beta}{h}) \end{aligned}$$

In other words,  $M_\alpha(\xi) \widehat{f}(\xi - \frac{\beta}{h})$  can be written as a product of tensor product multipliers applied to  $\widehat{f}(\xi - \frac{\beta}{h})$ , namely

$$M_\alpha(\xi) \widehat{f}(\xi - \frac{\beta}{h}) = [\sigma_1(\xi)] \times \left[ \prod_{j=1}^{n_2} (\xi_{K_j})^{\alpha_{K_j}} m_h(\xi_{K_j}) \right] \times [\sigma_2(\xi)] \times \widehat{f}(\xi - \frac{\beta}{h}),$$

We have written  $\sigma_1(\xi) := \sigma_{1,\alpha,\beta}(\xi) := \prod_{j=1}^{n_1} (\xi_{J_j})^{\alpha_{J_j}} \varphi(h(\xi_{J_j} - \frac{\beta_{J_j}}{h})) m_h(\xi_{J_j}) = \prod_{j=1}^{n_1} \tau_1(\xi_{J_j})$ . In a similar way, we identify the factor where  $\beta = 0$  as  $\prod_{j=1}^{n_2} (\xi_{K_j})^{\alpha_{K_j}} m_h(\xi_{K_j})$  and the factor where  $|\beta| = 2\pi$  as  $\sigma_2(\xi) := \sigma_{2,\alpha,\beta}(\xi) := \prod_{j=1}^{n_3} (\xi_{L_j})^{\alpha_{L_j}} \varphi(h(\xi_{L_j} - \frac{\beta_{L_j}}{h})) m_h(\xi_{L_j})$ . It follows that

$$\begin{aligned} \|G_{\alpha,\beta}\|_p &= \left\| \left( M_\alpha \widehat{f}(\cdot - \beta/h) \right)^\vee \right\|_p \\ &\leq \| \sigma_{1,\alpha,\beta} \|_{\mathcal{M}_p} \times \| m_h \|_{\mathcal{M}_p}^{n_2} \times \left\| \left( \sigma_{2,\alpha,\beta} \times (\cdot)^{\alpha_K} \widehat{f}(\cdot - \beta/h) \right)^\vee \right\|_{L_p}. \end{aligned} \quad (5.5)$$

where we write  $(\cdot)^{\alpha_K} : \xi \mapsto \xi^{\alpha_K} := \prod_{j=1}^{n_2} \xi^{\alpha_{K_j}}$ .

**Estimating  $\|(\sigma_{1,\alpha,\beta})^\vee\|_{\mathcal{M}_p}$ :** We use the rough estimate  $\|(\sigma_{1,\alpha,\beta})^\vee\|_{\mathcal{M}_p} \leq \|(\sigma_{1,\alpha,\beta})^\vee\|_1$ . The  $L_1$  norm of each univariate factor  $\tau_1(\xi_{J_j}) := (\xi_{J_j})^{\alpha_{J_j}} \varphi(h(\xi_{J_j} - \frac{\beta_{J_j}}{h})) m_h(\xi_{J_j})$  is bounded by Claim 5.1, from which we obtain

$$\|(\sigma_1)^\vee\|_1 \leq \frac{C}{h^{n_1}} \prod_{j=1}^{n_1} \left| \frac{\beta_{J_j}}{h} \right|^{\alpha_{J_j}+2} e^{-c|\beta_{J_j}|^2/h^2}. \quad (5.6)$$

**Estimating  $\left\| \left( \sigma_{2,\alpha,\beta} \times (\cdot)^{\alpha_K} \widehat{f}(\cdot - \beta/h) \right)^\vee \right\|_{L_p}$ :** There is an immediate decomposition of  $\sigma_2(\xi)$  into

$$\prod_{j=1}^{n_3} (\tau_2(\xi_{L_j}) + \tau_3(\xi_{L_j})).$$

Thus Claims 5.2 and 5.3, and the fact that  $\left\| \left( \prod_{j=1}^{n_3} (\xi_{L_j} - \beta_{L_j})^{\alpha_{L_j}} \times \xi^{\alpha_K} \times \widehat{f}(\xi - \beta/h) \right)^\vee \right\|_p \leq \|f\|_{W_p^k}$  imply that

$$\left\| \left( \sigma_2 \times (\cdot)^{\alpha_K} \widehat{f}(\cdot - \beta/h) \right)^\vee \right\|_{L_p} \leq C \left[ \prod_{j=1}^{n_3} (1 + \|m_h\|_{\mathcal{M}_p}) \right] \|f\|_{W_p^k}. \quad (5.7)$$

with constant  $C$  depending on  $n$  and  $\alpha$ .

Applying estimates (5.6) and (5.7), which control  $\|(\sigma_{1,\alpha,\beta})^\vee\|_{\mathcal{M}_p}$ , and  $\left\| \left( \sigma_{2,\alpha,\beta} \times (\cdot)^{\alpha_K} \widehat{f}(\cdot - \beta/h) \right)^\vee \right\|_{L_p}$ , respectively, to (5.5), we bound the sum of the  $\|G_{\alpha,\beta}\|_p$ 's:

$$\sum_{\beta \in 2\pi\mathbb{Z}^n} \|G_{\alpha,\beta}\|_p \leq C(1 + \|m_h\|_{\mathcal{M}_p})^n \|f\|_{W_p^k} \left[ \prod_{j=1}^n \left( 3 + 2 \sum_{\ell=2}^{\infty} \left| \frac{2\pi\ell}{h} \right|^{\alpha_j+2} e^{-c|\frac{2\pi\ell}{h}|^2} \right) \right] \quad (5.8)$$

and the Lemma follows from Lemma 4.2. □

**Lemma 5.5.** *Suppose that  $\epsilon > 0$  and that  $\phi$  is a  $C^\infty$  function with support in  $B(0, \pi + \epsilon)$ . If  $\beta \in 2\pi\mathbb{Z}$ ,  $\beta > 2\pi$  and  $0 < h < 1$  then there exist constants  $c, C > 0$ , depending only on  $\epsilon$  and  $k$  so that*

$$\int_{\mathbb{R}} \left| \frac{d^2}{d\xi^2} \left[ \xi^k \phi\left(h\left(\xi - \frac{\beta}{h}\right)\right) m_h(\xi) \right] \right| d\xi \leq Ch^{-1} \left( \frac{|\beta|}{h} \right)^{2+k} \exp \left( -c \left| \frac{\beta}{h} \right|^2 \right). \quad (5.9)$$

For  $\beta \in 2\pi\mathbb{Z}$ ,  $|\beta| = 2\pi$  and  $\phi$  satisfying the extra condition

$$\text{supp}(\phi(\cdot - \beta)) \cap B(0, \pi + \epsilon) = \emptyset, \quad (5.10)$$

there exist constants  $c, C > 0$ , depending only on  $\epsilon$  and  $k$  so that for  $0 < h < 1$

$$\int_{\mathbb{R}} \left| \frac{d^2}{d\xi^2} \left[ \xi^k \phi\left(h\left(\xi - \frac{\beta}{h}\right)\right) m_h(\xi) \right] \right| d\xi \leq Ch^{-(3+k)} \exp \left( - \left| \frac{c}{h} \right|^2 \right). \quad (5.11)$$



*Proof.* Let  $b = \pi + 2\epsilon$ . We prove this by making the estimate

$$\int_{\mathbb{R}} \left| \frac{d^2}{d\xi^2} \left[ \xi^k \phi\left(h\left(\xi - \frac{\beta}{h}\right)\right) m_h(\xi) \right] \right| d\xi \leq \left( \frac{2b}{h} \right) \max_{\xi \in [\frac{\beta-b}{h}, \frac{\beta+b}{h}]} \left| \frac{d^2}{d\xi^2} \left[ \xi^k \phi\left(h\left(\xi - \frac{\beta}{h}\right)\right) m_h(\xi) \right] \right|.$$

By applying the product rule to the expression being maximized, we obtain

$$\frac{d^2}{d\xi^2} \left[ \xi^k \phi\left(h\left(\xi - \frac{\beta}{h}\right)\right) m_h(\xi) \right] = \sum_{|\gamma|=2} C_\gamma \times \left( D^{\gamma_1} \xi^k \right) \times \left( D^{\gamma_2} \phi\left(h\left(\xi - \frac{\beta}{h}\right)\right) \right) \times (D^{\gamma_3} m_h(\xi)).$$

with  $C_\gamma = \frac{2!}{\gamma_1! \gamma_2! \gamma_3!}$ .

Without loss, it suffices to consider only components where both derivatives are on the third factor, i.e., those of the form  $\xi^k \phi\left(h\left(\xi - \frac{\beta}{h}\right)\right) \frac{d^2}{d\xi^2} [m_h(\xi)]$ , since the other terms are small compared to these. Indeed,  $|D^\gamma \xi^k| = C |\xi^{k-\gamma}| \leq C |\beta/h|^k$  for  $\xi \in [(|\beta| - b)/h, (|\beta| + b)/h]$  (with  $C$  depending only on  $k$ ), and  $\max_{\xi \in [\frac{\beta-b}{h}, \frac{\beta+b}{h}]} \left| D^\gamma \phi\left(h\left(\xi - \frac{\beta}{h}\right)\right) \right| = h^{|\gamma|} \max_{\xi \in [\frac{\beta-b}{h}, \frac{\beta+b}{h}]} \left| \phi\left(h\left(\xi - \frac{\beta}{h}\right)\right) \right| \leq Ch^{|\gamma|}$ , since  $\tau := D^\gamma \phi$  is a  $C^\infty$  function with the same support as  $\phi$ .

Hence,

$$\max_{\xi \in [\frac{\beta-b}{h}, \frac{\beta+b}{h}]} \left| \frac{d^2}{d\xi^2} \left[ \xi^k \phi\left(h\left(\xi - \frac{\beta}{h}\right)\right) m_h(\xi) \right] \right| \leq C \left| \frac{\beta}{h} \right|^k \max_{h\xi \in \text{supp}(\phi(\cdot - \beta))} |m_h''(\xi)|.$$

The result now follows directly from Lemma 4.1. Indeed, we have:

$$\max_{h\xi \in \text{supp}(\phi(\cdot - \beta))} |m_h''(\xi)| \leq C \left( \frac{|\beta| + b}{h} \right)^2 \max_{h\xi \in \text{supp}(\phi(\cdot - \beta))} m_h(\xi) \leq C \left( \frac{|\beta| + b}{h} \right)^2 \max_{h\xi \in \text{supp}(\phi(\cdot - \beta))} \frac{e^{-\frac{1}{4}|\xi|^2}}{2e^{-\frac{1}{4}|\frac{\pi}{h}|^2}}.$$

If (5.10) holds, then the expression being maximized can be controlled by  $\exp(-\frac{1}{4} \frac{(|\pi + \epsilon|^2 - \pi^2)}{h^2})$  and (5.11) follows.

On the other hand, if  $|\beta| \geq 4\pi$ , then  $h\xi \in \text{supp}(\phi(\cdot - \beta))$  implies that  $|\xi|^2 \leq \frac{(|\beta| - b)^2}{h^2}$  and  $\exp(-\frac{1}{4}|\xi|^2) \exp(\frac{1}{4}|\frac{\pi}{h}|^2) \leq \exp(-c\frac{|\beta|^2}{h^2})$ , from which (5.9) follows.  $\square$

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